

AN INFINITE HORIZON STOCHASTIC MAXIMUM PRINCIPLE FOR DISCOUNTED CONTROL PROBLEM WITH LIPSCHITZ COEFFICIENTS

VIRGINIE KONLACK SOCGNIA AND OLIVIER MENOUKEU-PAMEN

ABSTRACT. In the present work, a stochastic maximum principle for discounted control of a certain class of degenerate diffusion processes with global Lipschitz coefficient is investigated. The value function is given by a discounted performance functional, leading to a stochastic maximum principle of semi-couple forward-backward stochastic differential equation with non smooth coefficients. The proof is based on the approximation of the Lipschitz coefficients by smooth ones and the approximation of the infinite horizon adjoint process.

1. INTRODUCTION

Stochastic optimal control has been extensively studied in the past decades due to its applications to mathematical finance, insurance, economics, engineering, etc. There are two main techniques to solve stochastic optimal control: The dynamic programming and the stochastic maximum principle. For the dynamic programming, the reader may consult [11, 22] and references therein.

In this paper, we shall use stochastic maximum principle to solve an infinite horizon stochastic optimal control problem when the coefficients of the state process are non smooth. There have been many studies on stochastic maximum principle. Under smoothness of the coefficients of the state process, Kushner [15, 16] introduces the necessary stochastic maximum principle for a class of controls adapted to a fixed filtration. This work was extended to a more general setting in [4, 5, 6, 14] under the assumption that the diffusion coefficient is control-free. The maximum principle is given in terms of an adjoint equation which is solution to a backward stochastic differential equation (BSDE). The previous results were extended by Peng [19] in the case of a control-dependent diffusion coefficient. The maximum principle in this case is given in terms of a first order and second order adjoint equations. The latter are solutions to non-linear BSDE. Let mention also that the stochastic maximum principle was extended to system with jumps in [12].

In all the above mentioned work, it is assumed that the coefficient of the controlled process are smooth. However, it is possible to weaken the conditions on the coefficients. In [18], the author uses the Clark generalized gradient and stable convergence of probability measure to prove a finite horizon stochastic maximum principle when the coefficients are non smooth. Using the Krylov estimate, the authors in [1] prove a finite horizon stochastic maximum principle when the coefficient are Lipschitz with the diffusion coefficient been elliptic. The previous results was extended in the case of a degenerate diffusion coefficient in [2] (see also [3, 9].)

In this paper, we generalize the previous result in infinite horizon. More precisely, assuming that the state coefficients are Lipschitz (with the diffusion coefficient being degenerate), we establish an infinite horizon stochastic maximum principle for a discounted control problem. Since the value

Date: November 2013.

2010 Mathematics Subject Classification. 93E20, 60H, 65N30.

Key words and phrases. forward-backward stochastic differential equations, degenerate diffusion, stochastic maximum principle.

function is given by a discounted cost functional, it can be seen as a solution to a linear backward stochastic differential equation. With this observation, we also extend the above mentioned works on stochastic maximum principle for non-smooth coefficients, to a stochastic maximum principle for forward-backward systems. We use the technique of absolute continuity of probability measure (see [7]) to define a linearized version of the controlled process. We also defined a slightly different controlled process on an enlarged probability space with the initial condition been taken as a random variable. As for maximum principle for infinite horizon stochastic optimal control with smooth coefficients, the reader may consult [13] and [17] and the references therein.

The paper is organized as follows: In Section 2, we state the control problem and give some preliminary results. Section 3 is devoted to the study of the infinite horizon discounted control problem. This section also contains the main results of the paper. In the Appendix, we prove some needed results.

2. STATEMENT OF THE PROBLEM AND PRELIMINARY RESULT

2.1. Statement of the problem. In the following, we denote by (Ω, \mathcal{F}, P) a complete probability space. Let $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t}$ be the completion of the natural filtration generated by the Brownian motion $(B(t))_{t \geq 0}$, where \mathcal{F}_0 contains all the P -null sets of \mathcal{F} and $\mathcal{F}_\infty = \bigcup_{t \geq 0} \mathcal{F}_t$. It is a complete right continuous filtration. We denote $|\cdot|$ and $\|\cdot\|$ the Euclidean norms in \mathbb{R}^d and $\mathbb{R}^{d \times N}$, respectively. In the following, we define some space of processes.

Definition 2.1. Let $\alpha \in \mathbb{R}$, $p \geq 0$ and \mathcal{Y} be a Banach space with norm $\|\cdot\|_{\mathcal{Y}}$

- $L_{\mathbb{R}_+, \alpha}^p(\mathcal{Y})$ is the space of all \mathcal{F}_t -measurable random variables $X : \Omega \times \mathbb{R}_+ \rightarrow \mathcal{Y}$ such that

$$E[e^{\alpha t} \|X(t)\|_{\mathcal{Y}}^p] < \infty,$$

- $S_{\mathbb{R}_+, \alpha}^p(\mathcal{Y})$ is the space of all càdlàg, adapted processes $X : \Omega \times \mathbb{R}_+ \rightarrow \mathcal{Y}$ such that

$$E[e^{\alpha t} \sup_{0 \leq t} \|X(t)\|_{\mathcal{Y}}^p] < \infty,$$

- $H_{\mathbb{R}_+, \alpha}^p(\mathcal{Y})$ is the space of all predictable processes $X : \Omega \times \mathbb{R}_+ \rightarrow \mathcal{Y}$ such that

$$E \left[\int_0^\infty e^{\alpha t} \|X(t)\|_{\mathcal{Y}}^p dt \right] < \infty,$$

- we define $\mathcal{V}_\alpha = S_{\mathbb{R}_+, \alpha}^p(\mathcal{Y}) \times H_{\mathbb{R}_+, \alpha}^p(\mathcal{Y})$.

For notational simplicity, we will write $L_{\mathbb{R}_+, \alpha}^p$, $S_{\mathbb{R}_+, \alpha}^p$, $H_{\mathbb{R}_+, \alpha}^p$ instead of $L_{\mathbb{R}_+, \alpha}^p(\mathcal{Y})$, $S_{\mathbb{R}_+, \alpha}^p(\mathcal{Y})$, $H_{\mathbb{R}_+, \alpha}^p(\mathcal{Y})$, respectively.

We suppose that the state process $X(t) = X^{(u)}(t)$; $0 \leq t$, $\omega \in \Omega$ is a controlled diffusion process of the form:

$$\begin{cases} dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t))dB(t) \\ X(0) = x_0 \end{cases} \quad (2.1)$$

where $(B(t))_{t \geq 0}$ is a Brownian motion defined on the probability space (Ω, \mathcal{F}, P) . Here, the functions $b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{A} \rightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{A} \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ are given Borel measurable functions. The control process

$$u : [0, \infty) \times \Omega \rightarrow \mathbb{A}$$

where \mathbb{A} is a compact subset of \mathbb{R}^d , is an admissible control if (2.1) has a unique (strong) solution $X = X^{(u)}$ such that u is a measurable \mathbb{F} -adapted process. We shall denote the set of all admissible controls by \mathcal{U}_{ad} .

62 Suppose that we are given a (discounted) performance (cost) functional of the form

$$J(t, u) = E \left[\int_t^\infty e^{-\beta s} h(s, X(s), u(s)) ds \middle| \mathcal{F}_t \right], \beta \text{ big enough}, \quad (2.2)$$

63 where $h : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{A} \rightarrow \mathbb{R}$ is a Borel measurable function.

64 The optimal control problem we are dealing with is to find the maximizer of the performance
65 functional i.e., determine $u^* \in \mathcal{U}_{ad}$ such that

$$J(u^*) = \sup_{u \in \mathcal{U}_{ad}} J(u), \quad (2.3)$$

66 where $J(u) = J(0, u)$

67 **Remark 2.2.** It can be shown that $J_t(u)$ is solution of the following infinite horizon linear BSDE

$$dY(t) = -[h(t, X(t), u(t)) - \beta Y(t)] dt + Z(t) dB(t), t \geq 0. \quad (2.4)$$

68 Equations (2.1) and (2.4) form a semi-couple controlled forward-backward stochastic differential
69 equation (FBSDE) with infinite horizon and we put $J(0, u) = J(u)$. Such system has been quite studied
70 in the literature, see for e.g., [20, 21, 23].

71 Since the system (2.1)-(2.4) is semi-coupled, one can first solve the forward equation on \mathbb{R}_+ , and
72 then use the solution X of (2.1) to find the solution (Y, Z) to the BSDE (2.4).

73 In order to solve and give some estimates of solutions of the system (2.1)-(2.4), we shall assume
74 that the functions b, σ and h satisfy the following conditions:

75 (H1) $b(t, x, \cdot) : \mathbb{A} \rightarrow \mathbb{R}^n$ is continuous, $b(\cdot, x, u), \sigma(\cdot, x)$ and $f(\cdot, x, y, u)$ are progressively measurable
76 for $(x, u) \in \mathbb{R}^d \times \mathbb{A}, x \in \mathbb{R}^d$ and $(x, y, u) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{A}$.

77 (H2) There exists a positive deterministic function $\lambda_1(t)$ bounded by $M > 0$ satisfying

$$78 \int_0^\infty \lambda_1^2(t) dt < \infty \text{ and } \int_0^\infty \lambda_1^4(t) dt < \infty \text{ such that for every } (t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$$

$$|b(t, x_1, u) - b(t, x_2, u)| + \|\sigma(t, x_1) - \sigma(t, x_2)\| \leq \lambda_1(t) |x_1 - x_2|.$$

79 (H3) For any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, there exists $M_1 \in \mathbb{R}_+$ such that

$$80 |b(t, x, u)| \leq |b(t, 0, u)| + M_1(1 + |x|),$$

$$\|\sigma(t, x)\| \leq M_1(1 + |x|).$$

81 (H4) There exists a constant $\alpha \in \mathbb{R}_-$ such that $-2\beta + 8 < \alpha < -16M^4 - 10$, and

$$E \left[\int_0^\infty e^{\alpha t} (|b(t, 0, u)|^4 + \|\sigma(t, 0)\|^4 + |f(t, 0, 0, u)|^4) dt \right] < \infty.$$

82 (H5) For any $(t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$, there exists C such that

$$|h(t, x_1, u) - h(t, x_2, u)| \leq C|x_1 - x_2|.$$

83 It follows from the Radamacher theorem that since the coefficients of (2.1) are Lipschitz continuous
84 in the state variable, they admit weak derivatives (see [8]). Let b_x, σ_x^j be Borel measurable functions
85 such that

$$\frac{\partial b}{\partial x} = b_x(t, x, a) \text{ dx-a.e.,}$$

$$\frac{\partial \sigma^j}{\partial x} = \sigma_x^j(t, x) \text{ dx-a.e.,}$$

86 then they are bounded by the Lipschitz function $\lambda_1(t)$. We shall also assume that

87 (H6) $b_x(t, x, a)$ is continuous in a uniformly in (t, x) .

88 Moreover,

89 (H7) $h(t, \cdot, a)$ is continuously differentiable and $h(t, x, a)$ and $h_x(t, x, a)$ are continuous in a uni-
90 formly in (t, x) .

91 Let us now introduce the space

$$\mathbf{S} = \left\{ v \in L^2(\xi dx); \text{ s.t. } \frac{\partial v}{\partial x_j} \in L^2(\xi dx), j = 1, \dots, d \right\},$$

92 where ξ is a continuous positive function on \mathbb{R}^d satisfying $\int |x|^2 \xi(x) dx < \infty$; $\frac{\partial v}{\partial x_j}$ is the derivative of
93 v in the sense of distribution. \mathbf{S} is Endowed with the norm

$$\|v\|_{\mathbf{S}} = \left[\int v^2 \xi dx + \sum_{1 \leq j \leq d} \int \left(\frac{\partial v}{\partial x_j} \right)^2 \xi dx \right]^{1/2},$$

94 \mathbf{S} is a Hilbert space with $\mathbf{S} \subset \mathbf{H}_{loc}^1(\mathbb{R}^d)$.

95 **2.2. Preliminary results.** In this section, we shall give some preliminary results needed in the proof
96 of the main result (Theorem 3.2). Let d be the following metric on the space \mathcal{U}_{ad} of admissible
97 controls

$$d(u, v) = P \otimes e^{\alpha t} dt \{ (\omega, t) \in \Omega \times \mathbb{R}_+, u(\omega, t) \neq v(\omega, t) \}.$$

98 **Lemma 2.3.** (\mathcal{U}_{ad}, d) is a complete metric space.

99 *Proof.* See Appendix B □

100 In the next Lemma, we give an estimate of the solution to the infinite horizon SDE (2.1) and show
101 that the function $u \rightarrow X^u(t)$ is continuous.

102 **Lemma 2.4.**

103 1. Assume that (H1)-(H4) hold, then the SDE (2.1) admits a unique solution
104 $(X(t))_{t \geq 0} \in S_{\mathbb{R}_+, \alpha}^2 \cap H_{\mathbb{R}_+, \alpha}^4$ satisfying:

$$E \left[\int_0^\infty e^{\alpha t} |X(t)|^4 dt \right] < \infty. \quad (2.5)$$

2.

$$E \left[\sup_{t \geq 0} e^{\alpha t} |X_1(t) - X_2(t)|^2 \right] \leq c (d(u, v))^{1/4}. \quad (2.6)$$

105 3. For $u, v \in \mathcal{U}_{ad}$,

$$|J(u) - J(v)| \leq K^{1/4} (d(u, v))^{1/4}. \quad (2.7)$$

Equation (2.7) implies that the cost functional $J : (\mathcal{U}_{ad}, d) \rightarrow \mathbb{R}$ defined by

$$J(u) = E \left[\int_0^\infty e^{-\beta t} h(t, X(t), u(t)) dt \right]$$

106 is continuous.

107 *Proof.* 1. The proof of existence and uniqueness can be found in [21]. Next, applying the Itô formula,
 108 we get

$$\begin{aligned} d|X(t)|^4 &= 2|X(t)|^2 d|X(t)|^2 + d\langle X^2(t) \rangle_t \\ &= 4|X(t)|^2 \langle X(t), b(t, X(t), u(t)) \rangle dt + 2|X(t)|^2 \|\sigma(t, X(t))\|^2 dt \\ &\quad + 4\langle X(t), \sigma(t, X(t)) \rangle^2 dt + 4|X(t)|^2 \langle X(t), \sigma(t, X(t)) \rangle dB(t). \end{aligned}$$

109 We have that

$$d(e^{\alpha t} |X(t)|^4) = \alpha e^{\alpha t} |X(t)|^4 dt + e^{\alpha t} d|X(t)|^4.$$

110 Hence, integrating from 0 to t and taking expectation gives

$$\begin{aligned} E[e^{\alpha t} |X(t)|^4] &= E[X_0^4] + \alpha E\left[\int_0^t e^{\alpha s} |X(s)|^4 ds\right] + 4E\left[\int_0^t e^{\alpha s} |X(s)|^2 \langle X(s), b(s, X(s), u(s)) \rangle ds\right] \\ &\quad + 2E\left[\int_0^t e^{\alpha s} |X(s)|^2 \|\sigma(s, X(s))\|^2 ds\right] + 4E\left[\int_0^t \langle X(s), \sigma(s, X(s)) \rangle^2 ds\right] \\ &\leq E[X_0^4] + \alpha E\left[\int_0^t e^{\alpha s} |X(s)|^4 ds\right] + 2E\left[\int_0^t e^{\alpha s} |X(s)|^4 ds\right] \\ &\quad + 2E\left[\int_0^t e^{\alpha s} |X(s)|^2 |b(s, X(s), u(s))|^2 ds\right] + 6E\left[\int_0^t e^{\alpha s} |X(s)|^2 \|\sigma(s, X(s))\|^2 ds\right]. \end{aligned}$$

111 Using (H2), we get

$$\begin{aligned} E[e^{\alpha t} |X(t)|^4] &\leq E[X_0^4] + (\alpha + 2) E\left[\int_0^t e^{\alpha s} |X(s)|^4 ds\right] \\ &\quad + 4E\left[\int_0^t e^{\alpha s} |X(s)|^2 (|b(s, 0, u(s))|^2 + \lambda^2(s) |X(s)|^2) ds\right] \\ &\quad + 12E\left[\int_0^t e^{\alpha s} |X(s)|^2 (\|\sigma(s, 0)\|^2 + \lambda^2(s) |X(s)|^2) ds\right] \\ &\leq E[X_0^4] + (\alpha + 2) E\left[\int_0^t e^{\alpha s} |X(s)|^4 ds\right] \\ &\quad + 4E\left[\int_0^t e^{\alpha s} \left(\frac{|X(s)|^4}{2} + |b(s, 0, u(s))|^4 + M^4 |X(s)|^4\right) ds\right] \\ &\quad + 12E\left[\int_0^t e^{\alpha s} \left(\frac{|X(s)|^4}{2} + \|\sigma(s, 0)\|^4 + M^4 |X(s)|^4\right) ds\right] \\ &\leq E[X_0^4] + (\alpha + 10 + 16M^4) E\left[\int_0^t e^{\alpha s} |X(s)|^4 ds\right] \\ &\quad + E\left[\int_0^t e^{\alpha s} (4|b(s, 0, u(s))|^4 + 12\|\sigma(s, 0)\|^4) ds\right]. \end{aligned}$$

112 Using (H4) and since $E[e^{\alpha t} |X(t)|^4] \geq 0$, we get

$$-(\alpha + 16M^4 + 10) E\left[\int_0^t e^{\alpha t} |X(t)|^4\right] \leq E[X_0^4] + 8E\left[\int_0^t e^{\alpha s} (|b(s, 0, u(s))|^4 + \|\sigma(s, 0)\|^4) ds\right].$$

113 The result follows using (H4) and Fatou's Lemma.

114 2. Using integration by part and Itô formulas, we get

$$\begin{aligned}
e^{\alpha t} |X_1(t) - X_2(t)|^2 &= \int_0^t e^{\alpha s} d|X_1^u(s) - X_2^v(s)|^2 + \alpha \int_0^t e^{\alpha s} |X_1^u(s) - X_2^v(s)|^2 ds \\
&= 2 \int_0^t e^{\alpha s} \langle X_1(s) - X_2(s), b(s, X_1(s), u(s)) - b(s, X_2(s), v(s)) \rangle ds \\
&\quad + 2 \int_0^t e^{\alpha s} \langle X_1(s) - X_2(s), \sigma(s, X_1(s)) - \sigma(s, X_2(s)) \rangle dB(s) \\
&\quad + \int_0^t e^{\alpha s} \|\sigma(s, X_1(s)) - \sigma(s, X_2(s))\|^2 ds + \alpha \int_0^t e^{\alpha s} |X_1(s) - X_2(s)|^2 ds.
\end{aligned} \tag{2.8}$$

115 Taking the supremum, using Burkholder-Davis-Gundy and Hölder inequalities, for $T \in (0, \infty]$, we get
 116 since α is negative,

$$E \left[\sup_{0 \leq t \leq T} e^{\alpha t} |X_1(t) - X_2(t)|^2 \right] \leq 2I_1 + I_2 + I_3, \tag{2.9}$$

117 where

$$I_1 = E \left[\sup_{0 \leq t \leq T} \int_0^t e^{\alpha s} \langle X_1(s) - X_2(s), b(s, X_1(s), u(s)) - b(s, X_2(s), v(s)) \rangle ds \right], \tag{2.10}$$

$$I_2 = KE \left[\int_0^t e^{2\alpha s} |X_1(s) - X_2(s)|^2 \|\sigma(s, X_1(s)) - \sigma(s, X_2(s))\|^2 ds \right]^{1/2}, \tag{2.11}$$

$$I_3 = E \left[\sup_{0 \leq t \leq T} \int_0^t e^{\alpha s} \|\sigma(s, X_1(s)) - \sigma(s, X_2(s))\|^2 ds \right]. \tag{2.12}$$

118 It follows from (H2) that

$$I_3 \leq E \left[\int_0^T \lambda_1^2(s) \sup_{0 \leq s \leq T} e^{\alpha s} |X_1(s) - X_2(s)|^2 ds \right]. \tag{2.13}$$

119 Using (H2) and the Young inequality,

$$\begin{aligned}
I_2 &= KE \left[\int_0^t e^{2\alpha s} |X_1(s) - X_2(s)|^2 \|\sigma(s, X_1(s)) - \sigma(s, X_2(s))\|^2 ds \right]^{1/2} \\
&\leq KE \left[\sup_{0 \leq t \leq T} e^{\alpha t} |X_1(t) - X_2(t)| \left(\int_0^T e^{\alpha t} \|\sigma(s, X_1(s)) - \sigma(s, X_2(s))\|^2 ds \right)^{1/2} \right] \\
&\leq K\epsilon E \left[\sup_{0 \leq t \leq T} e^{\alpha t} |X_1(t) - X_2(t)|^2 \right] + \frac{K}{\epsilon} E \left[\int_0^T \lambda_1^2(s) e^{\alpha s} |X_1(s) - X_2(s)|^2 ds \right].
\end{aligned} \tag{2.14}$$

120 We also have

$$\begin{aligned}
I_1 &\leq E \left[\int_0^T e^{\alpha s} \langle X_1(s) - X_2(s), b(s, X_1(s), u(s)) - b(s, X_2(s), v(s)) \rangle ds \right] \\
&= I_{11} + I_{12},
\end{aligned}$$

121 where

$$I_{11} = E \left[\int_0^T e^{\alpha s} \langle X_1(s) - X_2(s), b(s, X_1(s), u(s)) - b(s, X_2(s), u(s)) \rangle ds \right] \tag{2.15}$$

$$I_{12} = E \left[\int_0^T e^{\alpha s} \langle X_1(s) - X_2(s), b(s, X_1(s), u(s)) - b(s, X_2(s), v(s)) \rangle \chi_{u \neq v} ds \right]. \tag{2.16}$$

122 Using (H2), we get

$$\begin{aligned}
I_{11} &= E \left[\int_0^T e^{\alpha s} \langle X_1(s) - X_2(s), b(s, X_1(s), u(s)) - b(s, X_2(s), u(s)) \rangle ds \right] \\
&\leq E \left[\int_0^T \lambda_1(s) e^{\alpha s} |X_1(s) - X_2(s)|^2 ds \right] \\
&\leq E \left[\int_0^T \lambda_1(s) \sup_{0 \leq s \leq T} e^{\alpha s} |X_1(s) - X_2(s)|^2 ds \right].
\end{aligned} \tag{2.17}$$

123 Using once more (H2), we get

$$\begin{aligned}
I_{12} &= E \left[\int_0^T e^{\alpha s} \langle X_1(s) - X_2(s), b(s, X_1(s), u(s)) - b(s, X_2(s), v(s)) \rangle \chi_{u \neq v} ds \right] \\
&= E \left[\int_0^T e^{\alpha s} \langle X_1(s) - X_2(s), b(s, X_1(s), u(s)) - b(s, 0, u(s)) + b(s, 0, u(s)) \rangle \chi_{u \neq v} ds \right] \\
&\quad + E \left[\int_0^T e^{\alpha s} \langle X_1(s) - X_2(s), -b(s, 0, v(s)) + b(s, 0, v(s)) - b(s, X_2(s), v(s)) \rangle \chi_{u \neq v} ds \right] \\
&= E \left[\int_0^T e^{\alpha s} \left\{ \langle X_1(s), b(s, X_1(s), u(s)) - b(s, 0, u(s)) \rangle \right. \right. \\
&\quad \left. \left. + \langle X_1(s), b(s, 0, u(s)) - b(s, 0, v(s)) \rangle \right\} \chi_{u \neq v} ds \right] \\
&\quad + E \left[\int_0^T e^{\alpha s} \left(\langle -X_2(s), b(s, X_1(s), u(s)) - b(s, 0, u(s)) \rangle \right. \right. \\
&\quad \left. \left. + \langle -X_2(s), b(s, 0, u(s)) - b(s, 0, v(s)) \rangle \right) \chi_{u \neq v} ds \right] \\
&\quad + E \left[\int_0^T e^{\alpha s} \left(\langle X_1(s), b(s, 0, v(s)) - b(s, X_2(s), v(s)) \rangle \right. \right. \\
&\quad \left. \left. + \langle -X_2(s), b(s, 0, v(s)) - b(s, X_2(s), v(s)) \rangle \right) \chi_{u \neq v} ds \right] \\
&\leq I_{12}^1 + I_{12}^2,
\end{aligned} \tag{2.18}$$

124 with

$$\begin{aligned}
I_{12}^1 &= E \left[\int_0^T e^{\alpha s} \left\{ \lambda_1(s) |X_1(s)|^2 + |X_1(s)| \left(|b(s, 0, u(s))| + |b(s, 0, v(s))| \right) \right. \right. \\
&\quad \left. \left. + |X_1(s)| \left(|b(s, 0, v(s))| + |b(s, X_2(s), v(s))| \right) \right\} \chi_{u \neq v} ds \right]
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
I_{12}^2 &= E \left[\int_0^T e^{\alpha s} \left\{ \lambda_1(s) |X_2(s)|^2 + |X_2(s)| \left(|b(s, 0, u(s))| + |b(s, 0, v(s))| \right) \right. \right. \\
&\quad \left. \left. + |X_2(s)| \left(|b(s, 0, u(s))| + |b(s, X_1(s), u(s))| \right) \right\} \chi_{u \neq v} ds \right]
\end{aligned} \tag{2.20}$$

125 Using (H3) and Hölder inequality, we get

$$\begin{aligned}
I_{12}^1 &\leq E \left[\int_0^T \lambda_1(s) e^{\alpha s} |X_1(s)|^2 \chi_{u \neq v} ds \right] + E \left[\int_0^T e^{\alpha s} |X_1(s)| \left(|b(s, 0, u(s))| + |b(s, 0, v(s))| \right) \chi_{u \neq v} ds \right] \\
&\quad + E \left[\int_0^T e^{\alpha s} |X_1(s)| \left(2|b(s, 0, v(s))| + M_1(1 + |X_2(s)|) \right) \chi_{u \neq v} ds \right]
\end{aligned}$$

126

$$\begin{aligned}
&\leq E \left[\int_0^T \lambda_1(s) e^{\alpha s} |X_1(s)|^4 ds \right]^{1/2} E \left[\int_0^T \lambda_1(s) e^{\alpha s} \chi_{u \neq v} ds \right]^{1/2} \\
&\quad + 2E \left[\int_0^T e^{\alpha s} |X_1(s)|^2 ds \right]^{1/2} E \left[\int_0^T e^{\alpha s} \left(|b(s, 0, u(s))|^2 + |b(s, 0, v(s))|^2 \right) \chi_{u \neq v} ds \right]^{1/2} \\
&\quad + 2E \left[\int_0^T e^{\alpha s} |X_1(s)|^2 ds \right]^{1/2} E \left[\int_0^T e^{\alpha s} \left(4|b(s, 0, v(s))|^2 + 2M_1^2(1 + |X_2(s)|)^2 \right) \chi_{u \neq v} ds \right]^{1/2} \\
&\leq E \left[\int_0^T \lambda_1(s) e^{\alpha s} |X_1(s)|^4 ds \right]^{1/2} E \left[\int_0^T \lambda_1^2(s) e^{\alpha s} ds \right]^{1/4} E \left[\int_0^T e^{\alpha s} \chi_{u \neq v} ds \right]^{1/4} \\
&\quad + 4E \left[\int_0^T e^{\alpha s} |X_1(s)|^2 ds \right]^{1/2} E \left[\int_0^T e^{\alpha s} \left(|b(s, 0, u(s))|^4 + |b(s, 0, v(s))|^4 \right) ds \right]^{1/4} \\
&\quad \times E \left[\int_0^T e^{\alpha s} \chi_{u \neq v} ds \right]^{1/4} \\
&\quad + 32E \left[\int_0^T e^{\alpha s} |X_1(s)|^2 ds \right]^{1/2} E \left[\int_0^T e^{\alpha s} |b(s, 0, v(s))|^4 ds \right]^{1/4} E \left[\int_0^T e^{\alpha s} \chi_{u \neq v} ds \right]^{1/4} \\
&\quad + 12M_1^2 E \left[\int_0^T e^{\alpha s} |X_1(s)|^2 ds \right]^{1/2} E \left[\int_0^T e^{\alpha s} (1 + |X_2(s)|)^4 ds \right]^{1/4} E \left[\int_0^T e^{\alpha s} \chi_{u \neq v} ds \right]^{1/4} \\
&\leq K_1(T) E \left[\int_0^T e^{\alpha s} \chi_{u \neq v} ds \right]^{1/4} \\
&\leq K_1(T) (d(u, v))^{1/4}. \tag{2.21}
\end{aligned}$$

127 In the same way, we can show that

$$I_{12}^2 \leq K_2(T) (d(u, v))^{1/4}. \tag{2.22}$$

128 Substituting (2.21) and (2.22) into (2.18), we get

$$I_{12} \leq (K_1(T) + K_2(T)) (d(u, v))^{1/4} \tag{2.23}$$

129 Choosing ε in (2.14) such that $K\varepsilon < 1$, we get combining (2.17) and (2.23) that

$$\begin{aligned}
E \left[\sup_{0 \leq t \leq T} e^{\alpha t} |X_1(t) - X_2(t)|^2 \right] &\leq \left(\frac{K}{\varepsilon} + 1 \right) E \left[\int_0^T \lambda_1^2(s) \sup_{0 \leq s \leq T} e^{\alpha s} |X_1(s) - X_2(s)|^2 ds \right] \\
&\quad + 2(K_1(T) + K_2(T)) (d(u, v))^{1/4} \tag{2.24}
\end{aligned}$$

130 Hence by the Gronwall Lemma, we get

$$\begin{aligned}
&E \left[\sup_{0 \leq t \leq T} e^{\alpha t} |X_1(t) - X_2(t)|^2 \right] \\
&\leq 2 \left(K_1(T) + K_2(T) \right) \left(\frac{K}{\varepsilon} + 1 \right) (d(u, v))^{1/4} \left\{ 1 + \int_0^T \lambda_1^2(t) \exp \left(\int_t^T \lambda_1^2(r) dr \right) dt \right\}. \tag{2.25}
\end{aligned}$$

131 Using (H2), (H4), (2.5) and Fatou's lemma, we get

$$2 \left(K_1(T) + K_2(T) \right) \left(\frac{K}{\varepsilon} + 1 \right) \left\{ 1 + \int_0^T \lambda_1^2(t) \exp \left(\int_t^T \lambda_1^2(r) dr \right) dt \right\} \rightarrow K_\infty < \infty, \text{ as } T \rightarrow \infty, \tag{2.26}$$

132 and hence,

$$E \left[\sup_{t \geq 0} e^{\alpha t} |X_1(t) - X_2(t)|^2 \right] \leq K_\infty (d(u, v))^{1/4}.$$

133 3. Let $\tau > t$, using integration by part formula and (H5), we have

$$\begin{aligned} & e^{\alpha t} |Y_1(t) - Y_2(t)|^2 + (\alpha + 2\beta - 8) e^{\alpha s} \int_t^\tau |Y_1(s) - Y_2(s)|^2 ds + \int_t^\tau e^{\alpha s} |Z_1(s) - Z_2(s)|^2 ds \\ & \leq 12 \int_t^\tau e^{\alpha s} \left(|X_1(s)|^2 + |X_2(s)|^2 + |h(s, 0, u(s))|^2 + |h(s, 0, v(s))|^2 \right) \chi_{u \neq v} ds \\ & + 2 \int_t^\tau e^{\alpha s} \langle Y_1(s) - Y_2(s), Z_1(s) - Z_2(s) \rangle dB(s). \end{aligned} \quad (2.27)$$

134 Taking conditional expectation with respect to \mathcal{F}_t on both sides of (2.27) and using the fact that
135 $\alpha + 2\beta - 8 \geq 0$, we get

$$\begin{aligned} & E \left[e^{\alpha t} |Y_1(t) - Y_2(t)|^2 \middle| \mathcal{F}_t \right] \\ & \leq 12 E \left[\int_t^\tau e^{\alpha s} \left(|X_1(s)|^2 + |X_2(s)|^2 + |h(s, 0, u(s))|^2 + |h(s, 0, v(s))|^2 \right) \chi_{u \neq v} ds \middle| \mathcal{F}_t \right], \end{aligned} \quad (2.28)$$

136 that is

$$e^{\alpha t} |Y_1(t) - Y_2(t)|^2 \leq 12 E \left[\int_t^\tau e^{\alpha s} \left(|X_1(s)|^2 + |X_2(s)|^2 + |h(s, 0, u(s))|^2 + |h(s, 0, v(s))|^2 \right) \chi_{u \neq v} ds \middle| \mathcal{F}_t \right].$$

137 Taking the supremum and squaring both sides, we have

$$\sup_{l \leq t \leq \tau} e^{2\alpha t} |Y_1(t) - Y_2(t)|^4 \leq 144 \left(\sup_{l \leq t \leq \tau} M_{\mathcal{F}_t} \right)^2, \quad (2.29)$$

138 where

$$M_{\mathcal{F}_t} = E \left[\int_l^\tau e^{\alpha s} \left(|X_1(s)|^2 + |X_2(s)|^2 + |h(s, 0, u(s))|^2 + |h(s, 0, v(s))|^2 \right) \chi_{u \neq v} ds \middle| \mathcal{F}_t \right]. \quad (2.30)$$

139 Note that $M_{\mathcal{F}_t}$ is a right continuous martingale on $[0, \tau]$ with terminal condition

$$E \left[\int_l^\tau e^{\alpha s} \left(|X_1(s)|^2 + |X_2(s)|^2 + |h(s, 0, u(s))|^2 + |h(s, 0, v(s))|^2 \right) \chi_{u \neq v} ds \right].$$

140 Thus, it follows from the Doob's maximal inequality and Hölder inequality that

$$\begin{aligned} & E \left[\left(\sup_{l \leq t \leq \tau} M_{\mathcal{F}_t} \right)^2 \right] \\ & \leq 4 E \left[\int_l^\tau e^{\alpha s} \left(|X_1(s)|^2 + |X_2(s)|^2 + |h(s, 0, u(s))|^2 + |h(s, 0, v(s))|^2 \right) \chi_{u \neq v} ds \right]^2 \\ & \leq 16 E \left[\int_l^\tau e^{\alpha s} \left(|X_1(s)|^4 + |X_2(s)|^4 + |h(s, 0, u(s))|^4 + |h(s, 0, v(s))|^4 \right) ds \right] E \left[\int_l^\tau e^{\alpha s} \chi_{u \neq v} ds \right] \end{aligned} \quad (2.31)$$

141 Hence taking expectation in (2.29) and combining with (2.31), we get

$$\begin{aligned} & E \left[\sup_{0 \leq t \leq \tau} e^{2\alpha t} |Y_1(t) - Y_2(t)|^4 \right] \\ & \leq 2304 E \left[\int_0^\tau e^{\alpha s} \left(|X_1(s)|^4 + |X_2(s)|^4 + |h(s, 0, u(s))|^4 + |h(s, 0, v(s))|^4 \right) ds \right] \\ & \times E \left[\int_0^\tau e^{\alpha s} \chi_{u \neq v} ds \right]. \end{aligned} \quad (2.32)$$

142 We get

$$\begin{aligned}
|Y_1(0) - Y_2(0)|^4 &= E \left[|Y_1(0) - Y_2(0)|^4 \right] \\
&\leq E \left[\sup_{0 \leq t \leq \tau} e^{2\alpha t} |Y_1(t) - Y_2(t)|^4 \right] \\
&\leq 2304 E \left[\int_0^\tau e^{\alpha s} \left(|X_1(s)|^4 + |X_2(s)|^4 + |h(s, 0, u(s))|^4 + |h(s, 0, v(s))|^4 \right) ds \right] \\
&\quad \times E \left[\int_0^\tau e^{\alpha s} \chi_{u \neq v} ds \right]. \tag{2.33}
\end{aligned}$$

143 Letting $\tau \rightarrow \infty$, using Fatou's lemma, (H4) and (2.5), we get $|Y_1(0) - Y_2(0)|^4 \leq Kd(u, v)$. Therefore,

144 $|J(u) - J(v)| \leq K^{1/4} \left(d(u, v) \right)^{1/4}.$

145 □

146 3. MAIN RESULT

147 In this section, we shall state and prove the main result. Define the enlarged probability space
 148 $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ by $\tilde{\Omega} := \mathbb{R}^d \times \Omega$, $\tilde{\mathcal{F}}$ is the Borel σ -field over $\tilde{\Omega}$ and $\tilde{P} := \xi dx \otimes P$. Define the Brownian
 149 motion $\{\tilde{B}(t)\}_{t \geq 0}$ by $\tilde{B}(t, x, \omega) = B(t, \omega)$. Let \tilde{X} be the solution to the following SDE

$$\begin{cases} d\tilde{X}(t) = b(t, \tilde{X}(t), \tilde{u}(t))dt + \sigma(t, \tilde{X}(t))d\tilde{B}(t) \\ \tilde{X}(0) = x_0 \end{cases} \tag{3.1}$$

150 associated to the control $\tilde{u}(t, x, \omega) = u(t, \omega)$ on the enlarged filtered probability space
 151 $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t\}_{t \geq 0}, \tilde{P})$, where $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t\}_{t \geq 0}$ is the filtration generated by the Brownian motion
 152 $\{\tilde{B}(t)\}_{t \geq 0}$ augmented with \tilde{P} -null sets.

153 Assumptions (H2) and (H3) imply that (3.1) has a unique strong solution which is $\tilde{\mathbb{F}}$ -adapted. The
 154 uniqueness of the solution of (3.1) is slightly weaker than that of (2.1). This will enable us to perform
 155 computations for $(\tilde{X}(t))$ which are not defined for $(X(t))$. On the other hand, the uniqueness of the
 156 solution of (3.1) implies that $\forall t \geq 0, \tilde{X}(t) = X(t)$, \tilde{P} -a.s.

157 Recall that the aim is to find $u^* \in \mathcal{U}_{ad}$ such that the supremum in (2.3) is attained in u^* .

158 Let us define the Hamiltonian $H : [0, \infty) \times \mathbb{R}^d \times U \times \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$H(t, x, u, p) = \langle b(t, x, u), p \rangle + h(t, x, u) - \beta \langle x, p \rangle, \tag{3.2}$$

159 and define the corresponding $\tilde{\mathbb{F}}$ -adapted adjoint process $p(t) \in \mathbb{R}^d$ by

$$p(t) = \tilde{E}_t \left[\int_t^\infty \phi^*(s, t) \cdot e^{-\beta s} h_x(s, X(s), u(s)) ds \right], \tag{3.3}$$

160 where $(\phi(s, t))_{s \geq t \geq 0}$ is the fundamental solution of the linear equation

$$\begin{cases} d\phi(s, t) = b_x(s, \hat{X}(s), \hat{u}(s)) \cdot \phi(s, t) ds + \sum_{j \leq d} \sigma_x^j(s, \hat{X}(s), \hat{u}(s)) \cdot \phi(s, t) d\tilde{B}^j(s) \\ \phi(t, t) = I_d. \end{cases} \tag{3.4}$$

161 Here ϕ^* denotes the transpose of the matrix ϕ and \tilde{E}_t is the conditional expectation with respect to the
 162 σ -algebra $\tilde{\mathcal{F}}_t$, $t \geq 0$.

163 **Remark 3.1.** *It is worth pointing out that since $\tilde{\mathbb{F}}$ is a Brownian filtration, one can show that $(p(t))_{t \geq 0}$*
 164 *in (3.12) is solution to the linear BSDE*

$$\begin{cases} dp(t) = \{(b_x(t, x, u) - \beta)p + \sigma_x(t, x)q + h_x(t, x, u)\} dt + qd\tilde{B}(t) \\ \lim_{T \rightarrow \infty} p(T) = 0. \end{cases} \quad (3.5)$$

165 Our main result in this paper is the following theorem:

166 **Theorem 3.2** (Infinite horizon stochastic maximum principle). *Let assumptions (H1)-(H7) hold. Let*
 167 *(\hat{u}, \hat{X}) be an optimal pair of (2.1) and (2.2). Then there exists a $\tilde{\mathbb{F}}$ -adapted stochastic process $(\hat{p}(t))_{t \geq 0}$*
 168 *solution to (3.3) with ϕ given by (3.4), such that*

169 • (The maximum condition)

$$H(t, \hat{X}(t), \hat{u}(t), \hat{p}) = \max_{a \in \mathbb{A}} H(t, \hat{X}(t), a, \hat{p}), \text{ ds-a.e., } \tilde{P}\text{-a.s.} \quad (3.6)$$

170 To prove Theorem 3.2, we follow a procedure consisting of several steps (compare [2, 9]). We
 171 first approximate the coefficients of the state process by smooth coefficients (this was done in Section
 172 2.2). For each approximating coefficient, we define a perturbed controlled problem in an enlarged
 173 filtration. Note that the perturbed controlled problem has a solution, since the coefficient of the state
 174 process satisfied conditions for existence and uniqueness of solution of SDE. The solution to the
 175 original problem is then given as a limit of the perturbed ones.

176 Next we shall define the approximation sequence $(b^n)_{n \geq 0}$ and $(\sigma^n)_{n \geq 0}$ of the coefficients b and σ of
 177 the SDE (2.1). After, we shall consider a perturbed (\mathcal{P}^n) control problem on the enlarged probability
 178 space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, obtained by replacing b and σ in (2.1) by b^n and σ^n respectively.

179 Let consider a mollifier φ , that is φ is a non-negative C^∞ -function on \mathbb{R}^d with compact support in
 180 the unit ball such that $\int \varphi(y)dy = 1$. $(b^n)_{n \geq 0}$ and $(\sigma^n)_{n \geq 0}$ are defined by convolution i.e.,

$$b^n(t, x, a) = n \int b(t, x - y, a) \varphi(ny) dy \quad (3.7)$$

$$\sigma^n(t, x, a) = n \int \sigma(t, x - y, a) \varphi(ny) dy \quad (3.8)$$

181 Note that the functions $b^n(t, x, a)$ and $\sigma^n(t, x, a)$ are Borel measurable functions satisfying the follow-
 182 ing properties:

183 **Lemma 3.3.**

184 1. *The functions b^n and σ^n satisfy conditions (H2) and (H3) and*
 185 *there exists a positive constant $M \in \mathbb{R}_+$ such that*

$$|b^n(t, x, a) - b(t, x, a)| + |\sigma^n(t, x, a) - \sigma(t, x, a)| < \frac{M}{n} = \varepsilon_n, \quad t \in [0, \infty].$$

186 2. *The functions b^n and σ^n are C^1 -functions, and for all $t \in [0, \infty]$*

$$\lim_{n \rightarrow \infty} b_x^n(t, x, a) = b_x(t, x, a) \text{ dx-a.e.}$$

$$\lim_{n \rightarrow \infty} \sigma_x^n(t, x, a) = \sigma_x(t, x, a) \text{ dx-a.e.}$$

188 3. *For any $p \geq 1$ and $R > 0$,*

$$\lim_{n \rightarrow \infty} \int \int_{B(0, R) \times [0, \infty]} e^{\alpha t} \sup_{a \in \mathbb{A}} |b_x^n(t, x, a) - b_x(t, x, a)|^p dx dt = 0.$$

189 *Proof.* See Appendix B □

190 For $n = 1, 2, \dots$, let $Y^n = A^n$ (respectively X^n, \hat{X}^n) be the unique strong solution of controlled SDE
 191 corresponding to $v = u$ (respectively u^n, \hat{u}^n .) Then Y^n is solution to the following stochastic differential
 192 equation

$$\begin{cases} dY^n(t) = b^n(t, Y^n(t), v(t))dt + \sigma^n(t, Y^n(t))d\tilde{B}(t) \\ Y^n(0) = y_0, \end{cases} \quad (3.9)$$

193 Let J^n be the corresponding cost functional. Then J^n is given by

$$J^n(t) = \tilde{E} \left[\int_t^\infty e^{-\beta s} h(s, Y^n(s), v(s)) ds \middle| \mathcal{F}_t \right]. \quad (3.10)$$

194 Moreover, let $(\phi^n(s, t), s \geq t)$ be the fundamental solution of the linear equation

$$\begin{cases} d\phi^n(s, t) = b_x^n(t, Y^n(t), v(t)) \cdot \phi^n(s, t) dt + \sum_{j \leq d} \sigma_x^{j,n}(t, Y^n(t)) \cdot \phi^n(s, t) d\tilde{B}^j(t) \\ \phi^n(t, t) = I_d, \end{cases} \quad (3.11)$$

195 where $Y^n = X^n, \hat{X}^n$ is the unique strong solution of controlled SDE corresponding to $v = u^n, \hat{u}^n$, re-
 196 spectively. Furthermore, for each (v, Y^n) , we define the associated Hamiltonian H^n and adjoint process
 197 as follows:

$$R^n(t) = \tilde{E}_t \left[\int_t^\infty \phi^{n,*}(s, t) \cdot e^{-\beta s} h_x(s, Y^n(s), v(s)) ds \right] \quad (3.12)$$

198 and

$$H^n(t, y, v, R) = \langle R, b^n(t, y, v) \rangle + h(y, v) - \beta \langle y, R \rangle, \quad (3.13)$$

199 where $\phi^{n,*}$ denotes the transpose of the matrix ϕ^n and the adjoint process $R^n = P^n$ or p^n are associated
 200 to (u^n, X^n) or (\hat{u}^n, \hat{X}^n) , respectively.

201 In the following lemma, we give the relations between the original control problem with the per-
 202 turbed ones.

203 **Lemma 3.4.** Assume that the conditions of Theorem 3.2 hold. Let $X(t)$ and $A^n(t)$ be the strong
 204 solutions of (2.1) and (3.9) respectively, corresponding to an admissible control u . Then we have

205 1. There exists a positive real number M_4 such that

$$\tilde{E} \left[\sup_{t \geq 0} e^{\alpha t} |X(t) - A^n(t)|^2 \right] \leq M_4 (\varepsilon_n)^2, \quad (3.14)$$

206 2. There exists a positive real number M_5 such that

$$|J^n(u) - J(u)| \leq M_5 \varepsilon_n, \quad (3.15)$$

207 with $\varepsilon_n = M/n$.

208 *Proof.* See Appendix B □

209 **Remark 3.5.** It is worth mentioning that if (\hat{X}, \hat{u}) is an optimal pair for our control problem (2.1) and
 210 (2.2), then \hat{u} is not automatically optimal for the perturbed control problem (\mathcal{P}^n) . But if we define
 211 $\delta_n = 2M_5 \varepsilon_n$, then it follows from Lemma 3.4 that

$$J^n(\hat{u}) \leq \inf \{ J^n(u), u \in \mathcal{U}_{ad} \} + \delta_n$$

212 This means \hat{u} is ε_n -optimal for the perturbed problem \mathcal{P}^n . From Lemma 2.3, the cost functional J^n is
 213 continuous with respect to the topology induced by the metric d , hence, using Ekeland's variational
 214 principle for \hat{u} with $v_n = \varepsilon_n^{1/2}$, there exists an admissible control u^n satisfying:

- 215 1. $d(u^n, \hat{u}) \leq \varepsilon_n^{1/2}$
 216 2. $J^n(u^n) \leq J^n(\hat{u})$
 217 3. u^n is optimal for the cost $J^n(u) + \varepsilon_n^{1/2} d(u, u^n)$

218 The next result gives an ‘‘Eckeland’s variational principle’’ type result satisfy by

219 $\tilde{E} \left[e^{-\beta t} H^n(t, X^n(t), u^n(t), P^n(t)) \right]$ and $\tilde{E} \left[e^{-\beta t} H^n(t, X^n(t), v', P^n(t)) \right], v' \in \mathbb{A}.$

220 **Proposition 3.6.** Assume that the conditions of Theorem 3.2 are satisfied. For each $\varepsilon_n > 0$, there
 221 exists an admissible control u^n and a $\tilde{\mathbb{F}}$ -adapted process $(P^n(t))$ given by (3.12) and a Lebesgue null
 222 set N such that for $t \notin N$

$$\tilde{E} \left[e^{-\beta t} H^n(t, X^n(t), u^n(t), P^n(t)) \right] \geq \tilde{E} \left[e^{-\beta t} H^n(t, X^n(t), v', P^n(t)) \right] - \varepsilon_n^{1/2} \quad (3.16)$$

223 for every \mathbb{A} -valued \mathcal{F}_t -measurable random variable v' .

224 *Proof.* See Appendix B □

225 **Corollary 3.7.** Let $\hat{X}^n(t)$ denote the unique strong solution of (3.9) corresponding to \hat{u} . Let $(\phi^n(s, t))_{s \geq t}$
 226 be the fundamental solution of the linear equation (3.11), $p^n(t)$ be the adjoint equation associated to
 227 the perturbed control problem given by (3.12) and $H^n(t, X^n(t), \hat{u}(t), p^n(t))$ be the Hamiltonian defined
 228 by (3.13). Then there exists an $\tilde{\mathbb{F}}$ -adapted process $(p^n(t))$ given by (3.12) and a Lebesgue null set N
 229 such that, for $t \notin N$

$$\tilde{E} \left[e^{-\beta t} H^n(t, \hat{X}^n(t), \hat{u}(t), p^n(t)) \right] \geq \tilde{E} \left[e^{-\beta t} H^n(t, \hat{X}^n(t), v', p^n(t)) \right] - \varepsilon_n^{1/3} \quad (3.17)$$

230 for every \mathbb{A} -valued \mathcal{F}_t -measurable random variable v' .

231 In the following result, we show that the sequence $(\phi^n(s, t))_{s \geq t}$ (respectively $(p^n(t))_{t \geq 0}$ and
 232 $H^n(t, \hat{X}^n(t), \hat{u}(t), p^n(t))$) defined by (3.11) (respectively (3.12) and (3.13)) associated to the controlled
 233 process $\hat{X}^n(t)$ given by (3.9), converge in $S_{\mathbb{R}^+, \alpha}^2$ norm to $(\phi(s, t))_{s \geq t}$ defined by (3.4) (respectively in
 234 $S_{\mathbb{R}^+, \alpha}^2$ and norm $L_{\mathbb{R}^+, \alpha}^2$ to $(p(t))_{t \geq 0}$ and $H(t, \hat{X}(t), \hat{u}(t), p(t))$).

235 **Proposition 3.8.** Assume that the conditions of Theorem 3.2 hold. Then we have

$$\lim_{n \rightarrow \infty} \tilde{E} \left[\sup_{t \geq 0} e^{\alpha t} |\phi^n(s, t) - \phi(s, t)|^2 \right] = 0, \quad (3.18)$$

$$\lim_{n \rightarrow \infty} \tilde{E} \left[\sup_{t \geq 0} e^{\alpha t} |p^n(t) - p(t)|^2 \right] = 0, \quad (3.19)$$

$$\lim_{n \rightarrow \infty} \tilde{E} \left[e^{\alpha t} |H^n(t, \hat{X}^n(t), \hat{u}(t), p^n(t)) - H(t, \hat{X}(t), \hat{u}(t), p(t))| \right] = 0. \quad (3.20)$$

238 *Proof.* See Appendix B □

Proof of Theorem 3.2. Applying Corollary 3.7 and Proposition 3.8, we have

$$\tilde{E} \left[H(t, \hat{X}(t), \hat{u}(t), p(t)) \right] \geq \tilde{E} \left[H(t, \hat{X}(t), v, p(t)) \right], \text{ dt-a.s.}$$

for every \mathbb{A} -valued \mathcal{F}_t random variable v . Let $a \in \mathbb{A}$ and define

$$\mu(t) := H(t, \hat{X}(t), \hat{u}(t), p(t)) - H(t, \hat{X}(t), v, p(t)).$$

239 Using (3.17), we have $\forall A_t \in \tilde{\mathcal{F}}_t, \tilde{E} \left[\chi_{A_t} \mu(t) \right] \geq 0$, dt – a.s. The result follows using the fact that $\mu(t)$
 240 is $\tilde{\mathcal{F}}_t$ -measurable and hence $\mu(t) \geq 0$, dt-a.s. □

ACKNOWLEDGMENT

The first author was supported by the NRF (National Research Foundation) grant number SFP 1208157898.

During this work, the second author visited the Advanced Mathematics of Finance programme of the University of the Witwatersrand. He thanks the school of Computational and Applied Mathematics for providing nice work environment and Prof Coenraad Labuschagne for his hospitality and encouragement.

APPENDIX A.

The following result due to Ekeland's provide a variational principle and can be found in [10].

Lemma A.1 (Ekeland's variational principle). *Let (V, d) be a complete metric space and $F : V \rightarrow \mathbb{R} \cup \{\infty\}$ be a lower-semi continuous and bounded function. Let $\varepsilon > 0$ and a point $u^\varepsilon \in V$ such that $F(u^\varepsilon) \leq \inf_V F + \varepsilon$. Then for any $\delta > 0$, there exists $v \in V$ such that the followings hold:*

1. $F(v) \leq F(u^\varepsilon)$;
2. $d(u^\varepsilon, v) \leq \delta$;
3. $\forall w \neq v; F(w) > F(v) - \frac{\varepsilon}{\delta} d(v, w)$

In order to formulate principle, we also need the following technical result which also can be found in [7].

Lemma A.2 (The Bouleau-Hirsch flow property). *Let \tilde{X} be the solution of the SDE (3.1) on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{P})$. Then for \tilde{P} a.e ω*

- (1) *For all $t \geq 0$, $g \rightarrow \tilde{X}(t, \omega)$ is in \mathcal{S}^d .*
- (2) *There exists a $\tilde{\mathcal{F}}_t$ -adapted $\mathbf{GL}(\mathbb{R})$ -value continuous process $(\tilde{\phi}(t))_{t \geq 0}$ such that for \tilde{P} -almost every ω and every $t \geq 0$, $\frac{\partial}{\partial g}(X^g(t, \omega)) = \tilde{\phi}_t(g, \omega)$ dx-a.e., where the differentiation is in the sense of distributions.*
- (3) *$\forall t \geq 0$, the image measure of \tilde{P} through the map $\tilde{X}(t)$ is absolutely continuous with respect to the Lebesgue measure.*
- (4) *The distributional derivative $\tilde{\phi}(t)$ is the fundamental solution of the linear stochastic differential equation*

$$\begin{cases} d\tilde{\phi}(s, t) = b_x(s, \hat{X}(s), \hat{u}(s))\tilde{\phi}(s, t)ds + \sum_{j \leq d} \sigma_x^j(s, \hat{X}(s), \hat{u}(s))\tilde{\phi}(s, t)d\tilde{B}^j(s), & s \geq t, \\ \tilde{\phi}(t, t) = I_d, \end{cases}$$

where b_x and σ_x^j are versions of the almost everywhere derivatives of b and σ^j respectively.

APPENDIX B. PROOF OF AUXILIARY RESULTS

Proof of Lemma 2.3. Let us first show that d is a distance. Let $u, v, w \in \mathcal{U}_{ad}$, and let define the following sets

$$\begin{aligned} A &:= \{(\omega, t) \in \Omega \times \mathbb{R}_+, u(\omega, t) \neq w(\omega, t)\}; \\ A_1 &:= \{(\omega, t) \in \Omega \times \mathbb{R}_+, u(\omega, t) \neq v(\omega, t)\}; \\ A_2 &:= \{(\omega, t) \in \Omega \times \mathbb{R}_+, v(\omega, t) \neq w(\omega, t)\}. \end{aligned}$$

272 Then $A \subset A_1 \cup A_2$, that is $P \otimes e^{\alpha t}(A) \leq P \otimes e^{\alpha t}(A_1 \cup A_2) \leq P \otimes e^{\alpha t}(A_1) + P \otimes e^{\alpha t}(A_2)$ and hence,
 273 $d(u, w) \leq d(u, v) + d(v, w)$

274 Next, let us show that (\mathcal{U}_{ad}, d) is complete. Let $(u_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in \mathcal{U}_{ad} . Then there
 275 exists a subsequence (u_{n_k}) such that $d(u_{n_k}, u_{n_{k+1}}) \leq \frac{1}{2^k}$. Define

$$A_k := \bigcup_{p \geq k} \{(\omega, t) \in \Omega \times \mathbb{R}_+, u_{n_p}(\omega, t) \neq u_{n_{p+1}}(\omega, t)\}$$

276 and

$$\mu_1 := P \otimes e^{\alpha t} \text{meas}\{(\omega, t) \in \Omega \times \mathbb{R}_+, u_1(\omega, t) \neq u_2(\omega, t)\} = P \times \mu_2.$$

277 We have

$$\begin{aligned} \mu_1(A_k) &\leq \sum_{p=k}^{\infty} \mu_1(\{(\omega, t) \in \Omega \times \mathbb{R}_+, u_{n_p}(\omega, t) \neq u_{n_{p+1}}(\omega, t)\}) \\ &= \sum_{p=k}^{\infty} d(u_{n_p}, u_{n_{p+1}}) \\ &\leq \sum_{p=k}^{\infty} \frac{1}{2^k} = 2^{1-k} \end{aligned}$$

278 and $A_{k+1} \subset A_k$ by construction. Now, put $A_k^\omega = \{t \in \mathbb{R}_+, (\omega, t) \in A_k\}$ and
 279 $A_k^t = \{\omega \in \Omega, (\omega, t) \in A_k\}$, then $\mu_2(A_k^\omega) = 0$ P -a.s., since $\mu_1(A_k) = 0$. Define $\bar{u}(t) := u_{n_k}(t)$ for $t \notin A_k^\omega$
 280 then u_{n_k} converges to \bar{u} P -a.s. Since the full sequence $(u_n)_{n \geq 0}$ has the Cauchy property, it converges
 281 also and hence (\mathcal{U}_{ad}, d) is complete. \square

282 *Proof of Lemma 3.3.*

283 1. Let us show that b^n satisfies (H2). Since b satisfies (H2) and φ is a mollifier, we have

$$\begin{aligned} |b^n(t, x_1, a) - b^n(t, x_2, a)| &= n \left| \int \left(b(t, x_1 - y, a) - b(t, x_2 - y, a) \right) \varphi(ny) dy \right| \\ &\leq n \lambda_1(t) |x_1 - x_2| \int \varphi(ny) dy = \lambda_1(t) |x_1 - x_2|. \end{aligned}$$

284 In the same way, we can show that σ^n satisfies (H2). Let us show that b^n satisfies (H3). Since b
 285 satisfies (H3) and the support of φ is in the unit ball, we have

$$\begin{aligned} |b^n(t, x, a)| &\leq n \int |b(t, x - y, a)| \varphi(ny) dy \\ &\leq n \int \varphi(ny) \left(|b(t, 0, a)| + M_1 (1 + |x - y|) \right) dy \\ &\leq |b(t, 0, a)| + M_1 \left(1 + |x| + n \int |y| \varphi(ny) dy \right) \\ &\leq |b(t, 0, a)| + M_1 \left(1 + |x| + n \int_{|y| < 1} \varphi(ny) dy \right) \\ &\leq |b(t, 0, a)| + 2M_1 (1 + |x|). \end{aligned}$$

We first write b as

$$b(t, x, a) = n \int b(t, x, a) \varphi(ny) dy.$$

286 Hence, Using (H2), we have

$$\begin{aligned} |b^n(t, x, a) - b(t, x, a)| &\leq n \int |b(t, x - y, a) - b(t, x, a)| \varphi(ny) dy \\ &\leq n \lambda_1(t) \int_{|y| < \frac{1}{n}} |y|^2 \varphi(ny) dy \\ &\leq \frac{\lambda_1(t)}{n} \leq \frac{M}{n}. \end{aligned}$$

287 The rest of 1. follows in the same way.

288

289 2. The first statement follows from the fact that φ belongs to C^1 , and b and σ are locally integrable
 290 since they satisfy (H2). Let now show that $\lim_{n \rightarrow \infty} b_x^n(t, x, a) = b_x(t, x, a)$ dx-a.e. Since b is differen-
 291 tiable almost everywhere, with the derivative bounded by $\lambda_1(t)$, and since φ is compactly supported,
 292 we can apply the Lebesgue dominated convergence theorem to get

$$b_x^n(t, x, a) = n \int b_x(t, x, a) \varphi(ny) dy. \quad (\text{B.1})$$

293 Note that is true (B.1) almost everywhere. Since $b_x(t, x, a)$ is uniformly continuous in (t, x) , as in the
 294 proof of 1., we get

$$b_x^n(t, x, a) - b_x(t, x, a) = n \int (b_x(t, x - y, a) - b_x(t, x, a)) \varphi(ny) dy.$$

295 The uniform continuity of b also implies that for $\varepsilon > 0$, there exists $\eta > 0$, such that,

296 $|y| < \eta \Rightarrow |b_x(t, x - y, a) - b_x(t, x, a)| < \varepsilon$ for all x . Hence if $\frac{1}{n} < \eta$, we get

$$|b_x(t, x - y, a) - b_x(t, x, a)| < \varepsilon \text{ dx-a.e.,}$$

297 meaning

$$\lim_{n \rightarrow \infty} b_x^n(t, x, a) = b_x(t, x, a) \text{ dx-a.e.}$$

298 3. Since b_x^n and b_x are bounded functions, then according to the Lebesgue dominated convergence
 299 theorem, it is enough to show that

$$\forall t \in \mathbb{R}_+, \sup_{a \in \mathbb{A}} |b_x^n(t, x, a) - b_x(t, x, a)| \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ dx-a.e.}$$

300 It follows from 2. that, for each fix $t \in \mathbb{R}_+, a \in \mathbb{A}$, there exists a dx-negligeable subset $N(a) \subset \mathbb{R}^d$ such
 301 that

$$\forall x \notin N(a), \sup_{a \in \mathbb{A}} |b_x^n(t, x, a) - b_x(t, x, a)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

302 Define $N := \bigcup_{a \in \mathbb{A}} N(a)$ and $\dot{A} := \{a \in A, a \text{ has rational component}\}$. Then by denseness argument,
 303 for $a \in A, \exists (a_p) \in \dot{A}$ such that $\lim_{p \rightarrow \infty} a_p = a$. Hence,

$$\begin{aligned} |b_x^n(t, x, a) - b_x(t, x, a)| &\leq |b_x^n(t, x, a) - b_x^n(t, x, a_p)| + |b_x^n(t, x, a_p) - b_x(t, x, a_p)| \\ &\quad + |b_x(t, x, a_p) - b_x(t, x, a)| \end{aligned} \quad (\text{B.2})$$

304 The first and the third terms in (B.2) converge to zero as p goes to $+\infty$ (since b_x^n and b_x are continuous
 305 in a) uniformly in (t, x) and n . The second term converges to 0 for each $x \notin N$. This achieves the
 306 proof. \square

307 *Proof of Lemma 3.4.* Let prove (3.14). Using Itô's product rule, we have

$$\begin{aligned} e^{\alpha t} |X(t) - A^n(t)|^2 &= 2 \int_0^t e^{\alpha s} \langle X(t) - A^n(t), b(s, X(s), u(s)) - b^n(s, A^n(s), u(s)) \rangle ds \\ &\quad + 2 \int_0^t e^{\alpha s} \langle X(t) - A^n(t), \sigma(s, X(s)) - \sigma^n(s, A^n(s)) \rangle d\tilde{B}(s) \\ &\quad + \int_0^t e^{\alpha s} \|\sigma(s, X(s)) - \sigma^n(s, A^n(s))\|^2 ds + \alpha \int_0^t e^{\alpha s} |X(s) - A^n(s)|^2 ds. \end{aligned}$$

308 Just as in the proof of Lemma 2.3, we have since $\alpha < 0$ that

$$\tilde{E} \left[\sup_{0 \leq t \leq T} e^{\alpha t} |X(t) - A^n(t)|^2 \right] \leq 2J_1 + J_2 + J_3, \quad (\text{B.3})$$

309 where

$$J_1 = \tilde{E} \left[\sup_{0 \leq t \leq T} \int_0^t e^{\alpha s} \langle X(t) - A^n(t), b(s, X(s), u(s)) - b^n(s, A^n(s), u(s)) \rangle ds \right], \quad (\text{B.4})$$

$$J_2 = K_2 \tilde{E} \left[\int_0^T e^{2\alpha s} |X(s) - A^n(s)|^2 \|\sigma(s, X(s)) - \sigma^n(s, A^n(s))\|^2 ds \right]^{1/2}, \quad (\text{B.5})$$

$$J_3 = \tilde{E} \left[\sup_{0 \leq t \leq T} \int_0^t e^{\alpha s} \|\sigma(s, X(s)) - \sigma^n(s, A^n(s))\|^2 ds \right]. \quad (\text{B.6})$$

310 From Lemma 3.3, we have

$$\begin{aligned} J_3 &\leq \tilde{E} \left[\sup_{0 \leq t \leq T} \int_0^t e^{\alpha s} (\|\sigma(s, X(s)) - \sigma^n(s, X(s))\|^2 + \|\sigma^n(s, X(s)) - \sigma^n(s, A^n(s))\|^2) ds \right] \\ &\leq K \frac{M}{n^2} + \tilde{E} \left[\sup_{0 \leq t \leq T} \int_0^t e^{\alpha s} \|\sigma^n(s, X(s)) - \sigma^n(s, A^n(s))\|^2 ds \right] \\ &\leq K \frac{M}{n^2} + \tilde{E} \left[\int_0^T \lambda_1^2(t) \sup_{0 \leq t \leq T} e^{\alpha t} |X(t) - A^n(t)|^2 dt \right]. \end{aligned} \quad (\text{B.7})$$

311 Using once more Lemma 3.3 and applying the Young inequality, we obtain

$$\begin{aligned} J_2 &= K_2 \tilde{E} \left[\int_0^T e^{2\alpha s} |X(s) - A^n(s)|^2 \|\sigma(s, X(s)) - \sigma^n(s, A^n(s))\|^2 ds \right]^{1/2} \\ &\leq K_2 \tilde{E} \left[\sup_{0 \leq t \leq T} e^{\alpha t/2} |X(t) - A^n(t)| \left(\int_0^T e^{\alpha t} \|\sigma(t, X(t)) - \sigma^n(t, A^n(t))\|^2 dt \right)^{1/2} \right] \\ &\leq K_2 \varepsilon \tilde{E} \left[\sup_{0 \leq t \leq T} e^{\alpha t} |X(t) - A^n(t)|^2 \right] \\ &\quad + \frac{K_2}{\varepsilon} \left(K \frac{M}{n^2} + \tilde{E} \left[\int_0^T \lambda_1^2(t) \sup_{0 \leq t \leq T} e^{\alpha t} |X(t) - A^n(t)|^2 dt \right] \right). \end{aligned} \quad (\text{B.8})$$

312 In the same way, using Lemma 3.3, Cauchy inequality and Young inequality, we get

$$\begin{aligned} J_1 &\leq K_3 \varepsilon \tilde{E} \left[\sup_{0 \leq t \leq T} e^{\alpha t} |X(t) - A^n(t)|^2 \right] \\ &\quad + \frac{K_3}{\varepsilon} \left(K \frac{M}{n^2} + \tilde{E} \left[\int_0^T \lambda_1(t) \sup_{0 \leq t \leq T} e^{\alpha t} |X(t) - A^n(t)|^2 dt \right] \right). \end{aligned} \quad (\text{B.9})$$

313 Choosing ε in (B.8) and (B.9) such that $K_2 \varepsilon + K_3 \varepsilon < 1$, combining (B.3)-(B.9) and using the Grown-
314 wall Lemma and the Fatou's Lemma, we obtain the desire result.

315 Let now show (3.15). Using (H4), we have

$$\begin{aligned} |J^n(u) - J(u)| &= \left| \tilde{E} \left[\int_0^\infty e^{-\beta t} \left(h(t, A^n(t), u(u)) - h(t, X(t), u(u)) \right) dt \right] \right| \\ &\leq \tilde{E} \left[\int_0^\infty e^{-\beta t} \left| h(t, A^n(t), u(u)) - h(t, X(t), u(u)) \right| dt \right] \\ &\leq C \tilde{E} \left[\int_0^\infty e^{-\beta t} |A^n(t) - X(t)| dt \right]. \end{aligned}$$

316 Using Hölder inequality, the fact that β is big enough and (3.14), we have

$$\begin{aligned} |J^n(u) - J(u)| &\leq C \left(\int_0^\infty e^{-\beta t/2} dt \right)^{1/2} \left(\tilde{E} \left[\int_0^\infty e^{-3\beta t/2} |A^n(t) - X(t)|^2 dt \right] \right)^{1/2} \\ &\leq C \left(\int_0^\infty e^{-\beta t/2} dt \right) \left(\tilde{E} \left[\sup_{t \geq 0} e^{-\beta t} |A^n(t) - X(t)|^2 \right] \right)^{1/2} \\ &\leq C \left(\int_0^\infty e^{-\beta t/2} dt \right) \left(\tilde{E} \left[\sup_{t \geq 0} e^{\alpha t} |A^n(t) - X(t)|^2 \right] \right)^{1/2} \\ &\leq C \left(\int_0^\infty e^{-\beta t/2} dt \right) \varepsilon_n. \end{aligned}$$

317 □

318 *Proof of Proposition 3.6.* By the Eckeland principle, with $\varepsilon = \varepsilon_n^{1/2}$, there is an admissible pair $(\tilde{X}^n, \tilde{u}^n)$
319 so that

$$d(u^n, \tilde{u}) \leq \varepsilon_n^{1/2}$$

and \tilde{u}^n is optimal for the cost functional J_ε^n defined by

$$J_\varepsilon^n(u(\cdot)) = J^n(u(\cdot)) + \varepsilon_n^{1/2} d(u^n, u).$$

320

321 This means that $(\tilde{X}^n, \tilde{u}^n)$ is an optimal pair for the system (3.9)-(3.10) with the new cost functional
322 J_ε^n . We shall now use the spike variation approach to derive a maximum principle for $(X^n(\cdot), u^n(\cdot))$.

323 Let $t_0 \in [s, \infty)$ and u , a fixed \mathbb{A} -valued \mathcal{F}_t -measurable random variable. For any $\delta > 0$, define
324 $u_\delta^n \in \mathcal{U}_{ad}[s, \infty)$ by:

$$u_\delta^n(t) = \begin{cases} u(t), & t \in [t_0, t_0 + \delta]; \\ \tilde{u}^n(t) & \text{otherwise.} \end{cases} \quad (\text{B.10})$$

325 Since $J_\varepsilon^n(\tilde{u}^n(\cdot)) \leq J_\varepsilon^n(\tilde{u}_\delta^n(\cdot))$ and $d(\tilde{u}_\delta^n(\cdot), \tilde{u}(\cdot)) \leq \int_{t_0}^{t_0+\delta} e^{\alpha t} dt \leq \delta$, we have that

$$\begin{aligned} 0 &\leq J_\varepsilon^n(u_\delta^n(\cdot)) - J_\varepsilon^n(\tilde{u}^n(\cdot)) \\ &= J^n(u_\delta^n(\cdot)) - J^n(u^n(\cdot)) + \varepsilon_n^{1/2} d(u_\delta^n(\cdot), u(\cdot)), \end{aligned}$$

326 and therefore

$$-\varepsilon_n^{1/2} \delta \leq J^n(u_\delta^n(\cdot)) - J^n(\tilde{u}^n(\cdot)). \quad (\text{B.11})$$

327 Define the process $V(t)$ by

$$V^n(t) := \frac{\partial}{\partial \delta} X_{u_\delta^n}^n \Big|_{\delta=0}. \quad (\text{B.12})$$

328 then

$$\begin{aligned} V^n(t) = & \mathbf{1}_{t \geq t_0} \left(b(t_0, \tilde{X}^n(t_0), \tilde{u}^n(t_0)) - b(t_0, \tilde{X}^n(t_0), u(t_0)) \right) \\ & + \int_{t_0}^t b_x^n(s, \tilde{X}^n(s), \tilde{u}^n(s)) V^n(s) ds + \int_{t_0}^t \sigma_x^n(s, \tilde{X}^n(s)) V^n(s) d\tilde{B}(s). \end{aligned} \quad (\text{B.13})$$

329 Moreover, we have

$$\begin{aligned} \frac{\partial}{\partial \delta} J^n(u_\delta^n) \Big|_{\delta=0} = & E \left[\int_{t_0}^\infty e^{-\beta s} h_x^n(s, \tilde{X}^n(s), \tilde{u}^n(s)) V^n(s) ds \right] \\ & + E \left[e^{-\beta t_0} \left(h^n(t_0, \tilde{X}^n(t_0), \tilde{u}^n(t_0)) - h^n(t_0, \tilde{X}^n(t_0), u(t_0)) \right) \right] \\ = & \lim_{T \rightarrow \infty} E \left[\int_{t_0}^T e^{-\beta s} h_x^n(s, \tilde{X}^n(s), \tilde{u}^n(s)) V^n(s) ds \right. \\ & \left. + e^{-\beta t_0} \left(h^n(t_0, \tilde{X}^n(t_0), \tilde{u}^n(t_0)) - h^n(t_0, \tilde{X}^n(t_0), u(t_0)) \right) \right]. \end{aligned} \quad (\text{B.14})$$

330 Put $P^n(T) = 0$ (see (3.5)). Hence, from Itô formula, (3.5) and (B.13), we have

$$\begin{aligned} 0 = & e^{-\beta T} P^n(T) V^n(T) \\ = & e^{-\beta t_0} P^n(t_0) \left(b^n(t_0, \tilde{X}^n(t_0), \tilde{u}^n(t_0)) - b^n(t_0, \tilde{X}^n(t_0), u(t_0)) \right) - \int_{t_0}^T \beta e^{-\beta s} P^n(s) V^n(s) ds \\ & + \int_{t_0}^T e^{-\beta s} P^n(s) dV^n(s) + \int_{t_0}^T e^{-\beta s} V^n(s) dP^n(s) + \int_{t_0}^T e^{-\beta s} d\langle V^n, P^n \rangle_s \\ = & e^{-\beta t_0} P^n(t_0) \left(b^n(t_0, \tilde{X}^n(t_0), \tilde{u}^n(t_0)) - b^n(t_0, \tilde{X}^n(t_0), u(t_0)) \right) - \int_{t_0}^T \beta e^{-\beta s} P^n(s) V^n(s) ds \\ & - \int_{t_0}^T e^{-\beta s} P^n(s) b_x^n(s, \tilde{X}^n(s), \tilde{u}^n(s)) V^n(s) ds + \int_{t_0}^T e^{-\beta s} P^n(s) \sigma_x^n(s, \tilde{X}^n(s)) V^n(s) d\tilde{B}(s) \\ & + \int_{t_0}^T e^{-\beta s} V^n(s) \left\{ \left(-b_x^n(s, \tilde{X}^n(s), \tilde{u}^n(s)) + \beta \right) P^n(s) - \sigma_x^n(s, \tilde{X}^n(s)) Q^n(s) - h_x(s, \tilde{X}^n(s), \tilde{u}^n(s)) \right\} ds \\ & + \int_{t_0}^T e^{-\beta s} V^n(s) Q^n(s) d\tilde{B}(s) + \int_{t_0}^T e^{-\beta s} \sigma_x^n(s, \tilde{X}^n(s)) V^n(s) Q^n(s) ds. \end{aligned} \quad (\text{B.15})$$

331 Taking expectation on both side of (B.15), we get

$$\begin{aligned} 0 = & E \left[e^{-\beta t_0} P^n(t_0) \left(b^n(t_0, \tilde{X}^n(t_0), \tilde{u}^n(t_0)) - b^n(t_0, \tilde{X}^n(t_0), u(t_0)) \right) \right] \\ & - E \left[\int_{t_0}^T e^{-\beta s} V^n(s) h_x^n(s, \tilde{X}^n(s), \tilde{u}^n(s)) ds \right]. \end{aligned}$$

332 Hence,

$$\begin{aligned} & E \left[\int_{t_0}^T e^{-\beta s} V^n(s) h_x^n(s, \tilde{X}^n(s), \tilde{u}^n(s)) ds \right] \\ = & E \left[e^{-\beta t_0} P^n(t_0) \left(b^n(t_0, \tilde{X}^n(t_0), \tilde{u}^n(t_0)) - b^n(t_0, \tilde{X}^n(t_0), u(t_0)) \right) \right]. \end{aligned} \quad (\text{B.16})$$

333 That is, taking the limit on the right hand side as T goes to ∞ will not change anything. Combining
 334 (B.11)-(B.16), we get

$$\begin{aligned} -\varepsilon_n^{1/2} &\leq E \left[P^n(t_0) \left(b^n(t_0, \tilde{X}^n(t_0), \tilde{u}^n(t_0)) - b^n(t_0, \tilde{X}^n(t_0), u(t_0)) \right) \right. \\ &\quad \left. + h^n(t_0, \tilde{X}^n(t_0), \tilde{u}^n(t_0)) - h^n(t_0, \tilde{X}^n(t_0), u(t_0)) \right] \\ &= E \left[e^{-\beta t_0} H^n(t_0, \tilde{X}^n(t_0), \tilde{u}^n(t_0), P^n(t_0)) - H^n(t_0, \tilde{X}^n(t_0), u(t_0), P^n(t_0)) \right]. \end{aligned}$$

335 Using (H1)-(H5) and the Hölder inequality, one can show that \tilde{X}^n and \tilde{u}^n can be replaced by X^n and
 336 u^n respectively and therefore, we get

$$\tilde{E} \left[e^{-\beta t} H^n(t, \hat{X}^n(t), u^n(t), P^n(t)) \right] \geq \tilde{E} \left[e^{-\beta t} H^n(t, \hat{X}^n(t), v, P^n(t)) \right] - \varepsilon_n^{1/2}.$$

337

□

338 *Proof of Proposition 3.8.* We shall only prove (3.18) and the rest is proved in the same way. Using
 339 Itô's product rule, we get

$$\begin{aligned} &e^{\alpha s} |\phi^n(s, t) - \phi(s, t)|^2 \\ &= 2 \int_0^s e^{\alpha r} \langle \phi^n(r, t) - \phi(r, t), b_x^n(r, \hat{X}^n(r), \hat{u}(r)) \cdot \phi^n(r, t) - b_x(r, \hat{X}(r), \hat{u}(r)) \cdot \phi(r, t) \rangle dr \\ &\quad + 2 \int_0^s e^{\alpha r} \langle \phi^n(r, t) - \phi(r, t), \sigma_x^n(r, \hat{X}^n(r)) \cdot \phi^n(r, t) - \sigma_x(r, \hat{X}(r)) \cdot \phi(r, t) \rangle d\tilde{B}(r) \\ &\quad + \int_0^s e^{\alpha r} \left\| \sigma_x^n(r, \hat{X}^n(r)) \cdot \phi^n(r, t) - \sigma_x(r, \hat{X}(r)) \cdot \phi(r, t) \right\|^2 dr + \alpha \int_0^s e^{\alpha r} |\phi^n(r, t) - \phi(r, t)|^2 dr. \end{aligned}$$

340 Using Burkholder-Davis-Grundy inequality and the fact that $\alpha < 0$, we have for $T > 0$

$$\tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s} |\phi^n(s, t) - \phi(s, t)|^2 \right] \leq 2L_1 + KL_2 + L_3,$$

341 where K is a positive constant and

$$L_1 = \tilde{E} \left[\int_t^T e^{\alpha r} \langle \phi^n(r, t) - \phi(r, t), b_x^n(r, \hat{X}^n(r), \hat{u}(r)) \cdot \phi^n(r, t) - b_x(r, \hat{X}(r), \hat{u}(r)) \cdot \phi(r, t) \rangle dr \right], \quad (\text{B.17})$$

$$L_2 = \tilde{E} \left[\int_t^T e^{\alpha r} \left| \phi^n(r, t) - \phi(r, t) \right|^2 \left\| \sigma_x^n(r, \hat{X}^n(r)) \cdot \phi^n(r, t) - \sigma_x(r, \hat{X}(r)) \cdot \phi(r, t) \right\|^2 dr \right]^{1/2}, \quad (\text{B.18})$$

$$L_3 = \tilde{E} \left[\int_t^T e^{\alpha r} \left\| \sigma_x^n(r, \hat{X}^n(r)) \cdot \phi^n(r, t) - \sigma_x(r, \hat{X}(r)) \cdot \phi(r, t) \right\|^2 dr \right]. \quad (\text{B.19})$$

342 Using Young inequality and (H2), we obtain

$$\begin{aligned} L_3 &\leq \tilde{E} \left[\int_t^T e^{\alpha r} \left\| \sigma_x^n(r, \hat{X}^n(r)) - \sigma_x(r, \hat{X}(r)) \right\|^2 |\phi^n(r, t)|^2 dr \right] \\ &\quad + \tilde{E} \left[\int_t^T e^{\alpha r} \left\| \sigma_x(r, \hat{X}(r)) \right\|^2 |\phi^n(r, t) - \phi(r, t)|^2 dr \right] \\ &\leq \frac{1}{2} \tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s} |\phi^n(s, t)|^4 \right] + \frac{1}{2} \tilde{E} \left[\int_t^T e^{\alpha r} \left\| \sigma_x^n(r, \hat{X}^n(r)) - \sigma_x(r, \hat{X}(r)) \right\|^4 dr \right] \\ &\quad + \tilde{E} \left[\int_t^T \lambda_1^2(r) \sup_{t \leq r \leq T} e^{\alpha r} |\phi^n(r, t) - \phi(r, t)|^2 dr \right]. \end{aligned} \quad (\text{B.20})$$

343 Using once more Young inequality and (H2), we get

$$\begin{aligned}
L_2 &\leq \varepsilon \tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s} |\phi^n(s, t) - \phi(s, t)|^2 \right] \\
&\quad + \frac{1}{\varepsilon} \tilde{E} \left[\int_t^T e^{\alpha r} \left\| \sigma_x^n(r, \hat{X}^n(r)) \cdot \phi^n(r, t) - \sigma_x(r, \hat{X}(r)) \cdot \phi(r, t) \right\|^2 dr \right] \\
&\leq \varepsilon \tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s} |\phi^n(s, t) - \phi(s, t)|^2 \right] + \frac{1}{\varepsilon} \tilde{E} \left[\int_t^T \lambda_1(r)^2 \sup_{t \leq r \leq T} e^{\alpha r} |\phi^n(r, t) - \phi(r, t)|^2 dr \right] \\
&\quad + \frac{1}{\varepsilon} \tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s/2} |\phi^n(s, t)|^2 \int_t^T e^{\alpha r/2} \left\| \sigma_x^n(r, \hat{X}^n(r)) - \sigma_x(r, \hat{X}(r)) \right\|^2 dr \right].
\end{aligned}$$

344 Applying Cauchy inequality to the last term and using the fact that $\alpha < 0$, we get

$$\begin{aligned}
L_2 &\leq \varepsilon \tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s} |\phi^n(s, t) - \phi(s, t)|^2 \right] + \frac{1}{\varepsilon} \tilde{E} \left[\int_t^T \lambda_1(r)^2 \sup_{t \leq r \leq T} e^{\alpha r} |\phi^n(r, t) - \phi(r, t)|^2 dr \right] \\
&\quad + \frac{M_1}{\varepsilon} \tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s} |\phi^n(s, t)|^4 \right]^{1/2} \tilde{E} \left[\int_t^T e^{\alpha r} \left\| \sigma_x^n(r, \hat{X}^n(r)) - \sigma_x(r, \hat{X}(r)) \right\|^4 dr \right]^{1/2}, \quad (\text{B.21})
\end{aligned}$$

345 where M_1 is a positive constant. Similarly as in (B.21), we get using Young inequality, (H2) and
 346 Cauchy inequality

$$\begin{aligned}
L_1 &\leq \tilde{E} \left[\int_t^T e^{\alpha r} |\phi^n(r, t) - \phi(r, t)| \left| b_x^n(r, \hat{X}^n(r), \hat{u}(r)) \cdot \phi^n(r, t) - b_x(r, \hat{X}(r), \hat{u}(r)) \cdot \phi(r, t) \right| dr \right] \\
&\leq \varepsilon' \tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s} |\phi^n(s, t) - \phi(s, t)|^2 \right] \\
&\quad + \frac{1}{\varepsilon'} \tilde{E} \left[\int_t^T e^{\alpha r} \left| b_x^n(r, \hat{X}^n(r), \hat{u}(r)) \cdot \phi^n(r, t) - b_x(r, \hat{X}(r), \hat{u}(r)) \cdot \phi(r, t) \right|^2 dr \right] \\
&\leq \varepsilon' \tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s} |\phi^n(s, t) - \phi(s, t)|^2 \right] + \frac{1}{\varepsilon'} \tilde{E} \left[\int_t^T \lambda_1^2(r) \sup_{t \leq r \leq T} e^{\alpha r} |\phi^n(r, t) - \phi(r, t)|^2 dr \right] \\
&\quad + \frac{M_2}{\varepsilon'} \tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s} |\phi^n(s, t)|^4 \right]^{1/2} \tilde{E} \left[\int_t^T e^{\alpha r} \left| b_x^n(r, \hat{X}^n(r), \hat{u}(r)) - b_x(r, \hat{X}(r), \hat{u}(r)) \right|^4 dr \right]^{1/2}, \quad (\text{B.22})
\end{aligned}$$

347 with M_2 is a positive constant. Choosing ε and ε' such that $K\varepsilon + \varepsilon' < 1$, and combining (B.20)-(B.22),
 348 we get

$$\begin{aligned}
&\tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s} |\phi^n(s, t) - \phi(s, t)|^2 \right] \\
&\leq M_\varepsilon \left(\tilde{E} \left[\int_t^T \lambda_1^2(r) \sup_{t \leq r \leq T} e^{\alpha r} |\phi^n(r, t) - \phi(r, t)|^2 dr \right] \right. \\
&\quad + \tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s} |\phi^n(s, t)|^4 \right]^{1/2} \tilde{E} \left[\int_t^T e^{\alpha r} \left\| \sigma_x^n(r, \hat{X}^n(r)) - \sigma_x(r, \hat{X}(r)) \right\|^4 dr \right]^{1/2} \\
&\quad \left. + 2 \tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s} |\phi^n(s, t)|^4 \right]^{1/2} \tilde{E} \left[\int_t^T e^{\alpha r} \left| b_x^n(r, \hat{X}^n(r), \hat{u}(r)) - b_x(r, \hat{X}(r), \hat{u}(r)) \right|^4 dr \right]^{1/2} \right),
\end{aligned}$$

349 where M_ε is a positive constant depending on M_1, M_2, ε and ε' . Using Gronwall's inequality, we get

$$\begin{aligned} & \tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s} |\phi^n(s, t) - \phi(s, t)|^2 \right] \\ & \leq M_\varepsilon(T) \tilde{E} \left[\sup_{t \leq s \leq T} e^{\alpha s} |\phi^n(s, t)|^4 \right]^{1/2} \left(\tilde{E} \left[\int_t^T e^{\alpha r} \left\| \sigma_x^n(r, \hat{X}^n(r)) - \sigma_x(r, \hat{X}(r)) \right\|^4 dr \right]^{1/2} \right. \\ & \quad \left. + \tilde{E} \left[\int_t^T e^{\alpha r} \left| b_x^n(r, \hat{X}^n(r), \hat{u}(r)) - b_x(r, \hat{X}(r), \hat{u}(r)) \right|^4 dr \right]^{1/2} \right), \end{aligned}$$

350 where $M_\varepsilon(T) = M_\varepsilon \int_t^T \lambda_1^2(r) dr$. As T goes to ∞ , we get by the Fatou's Lemma

$$\tilde{E} \left[\sup_{s \geq t} e^{\alpha s} |\phi^n(s, t) - \phi(s, t)|^2 \right] \leq M \tilde{E} \left[\sup_{s \geq t} e^{\alpha s} |\phi^n(s, t)|^4 \right]^{1/2} \left(I_n^{1/2} + J_n^{1/2} \right), \quad (\text{B.23})$$

351 with

$$I_n = \tilde{E} \left[\int_0^\infty e^{\alpha r} \left| b_x^n(r, \hat{X}^n(r), \hat{u}(r)) - b_x(r, \hat{X}(r), \hat{u}(r)) \right|^4 dr \right], \quad (\text{B.24})$$

$$J_n = \tilde{E} \left[\int_0^\infty e^{\alpha r} \left\| \sigma_x^n(r, \hat{X}^n(r)) - \sigma_x(r, \hat{X}(r)) \right\|^4 dr \right]. \quad (\text{B.25})$$

352 Let note that, just as in Lemma 2.4, one can show that $\tilde{E} \left[\sup_{s \geq t} e^{\alpha s} |\phi^n(s, t)|^4 \right] < \infty$. To show that the
353 limit (3.18) holds, it is enough to show that $I_n \rightarrow 0$ as $n \rightarrow \infty$ and $J_n \rightarrow 0$ as $n \rightarrow \infty$. We have

$$\begin{aligned} I_n & \leq M \tilde{E} \left[\int_0^\infty e^{\alpha r} \left| b_x^n(r, \hat{X}^n(r), \hat{u}(r)) - b_x(r, \hat{X}^n(r), \hat{u}(r)) \right|^4 dr \right] \\ & \quad + M \tilde{E} \left[\int_0^\infty e^{\alpha r} \left| b_x(r, \hat{X}^n(r), \hat{u}(r)) - b_x(r, \hat{X}(r), \hat{u}(r)) \right|^4 dr \right] \\ & \leq M \left(I_n^1 + I_n^2 \right), \end{aligned}$$

354 where

$$I_n^1 = \tilde{E} \left[\int_0^\infty \sup_{a \in \mathbb{A}} e^{\alpha r} \left| b_x^n(r, \hat{X}^n(r), a) - b_x(r, \hat{X}^n(r), a) \right|^4 dr \right], \quad (\text{B.26})$$

$$I_n^2 = \tilde{E} \left[\int_0^\infty \sup_{a \in \mathbb{A}} e^{\alpha r} \left| b_x(r, \hat{X}^n(r), a) - b_x(r, \hat{X}(r), a) \right|^4 dr \right]. \quad (\text{B.27})$$

355 Using the absolute continuity in the law of \hat{X}^n with respect to the Lesbesgue measure, and denoting
356 by $\varphi_t^n(y)$ its density, we get

$$I_n^1 = \int_0^\infty \int_{\mathbb{R}^d} \sup_{a \in \mathbb{A}} e^{\alpha t} \left| b_x^n(t, y, a) - b_x(t, y, a) \right|^4 \varphi_t^n(y) dy dt. \quad (\text{B.28})$$

357 For $t > 0$, let us compute limit

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \sup_{a \in \mathbb{A}} e^{\alpha t} \left| b_x^n(t, y, a) - b_x(t, y, a) \right|^4 \varphi_t^n(y) dy.$$

358 We know that $\tilde{E} \left[\sup_{t \geq 0} e^{\alpha t} |\hat{X}^n(t)|^4 \right] < \infty$ i.e., $\lim_{R \rightarrow \infty} \tilde{P} \left(\sup_{t \geq 0} e^{\alpha t} |\hat{X}^n(t)|^4 > R \right) = 0$. Using the
359 change of variable $y = e^{-\alpha t} z$, let us compute

$$\lim_{n \rightarrow \infty} \int_{B(0, R)} \sup_{a \in \mathbb{A}} \left| b_x^n(t, e^{-\alpha t} z, a) - b_x(t, e^{-\alpha t} z, a) \right|^4 \varphi_t^n(e^{-\alpha t} z) dz.$$

360 We know from 3. of Lemma 3.3 that

$$\sup_{a \in \mathbb{A}} \left| b_x^n(t, y, a) - b_x(t, y, a) \right|^4 \rightarrow 0 \text{ dy a.s at least for a sequence.}$$

361 It follows from Egorov's theorem that for every $\varepsilon > 0$, there exists a measurable set F with $\text{meas}(F) <$
 362 ε such that $\sup_{a \in \mathbb{A}} |b_x^n(t, y, a) - b_x(t, y, a)|$ converges uniformly to 0 on the set F^c . Hence we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{F^c} \sup_{a \in \mathbb{A}} \left| b_x^n(t, e^{-\alpha t} z, a) - b_x(t, e^{-\alpha t} z, a) \right|^4 \varphi_t^n(e^{-\alpha t} z) dz \\ & \leq \lim_{n \rightarrow \infty} \left(\sup_{F^c} \sup_{a \in \mathbb{A}} \left| b_x^n(t, e^{-\alpha t} z, a) - b_x(t, e^{-\alpha t} z, a) \right|^4 \right) \int_{\mathbb{R}^d} \varphi_t^n(e^{-\alpha t} z) dz \\ & = e^{\alpha t} \lim_{n \rightarrow \infty} \varepsilon_n \text{ where } \varepsilon_n \rightarrow 0. \end{aligned} \quad (\text{B.29})$$

363 On the other hand, using (H2), we have that

$$\begin{aligned} & \int_F \sup_{a \in \mathbb{A}} \left| b_x^n(t, e^{-\alpha t} z, a) - b_x(t, e^{-\alpha t} z, a) \right|^4 \varphi_t^n(e^{-\alpha t} z) dz \\ & \leq \tilde{E} \left[\sup_{a \in \mathbb{A}} \left| b_x^n(t, e^{-\alpha t} z, a) - b_x(t, e^{-\alpha t} z, a) \right|^4 \chi_{\{e^{-\alpha t} \hat{X}^n(t) \in F\}} \right] \\ & \leq 2\lambda_1^4(t) \tilde{P}(e^{-\alpha t} \hat{X}^n(t) \in F). \end{aligned} \quad (\text{B.30})$$

364 $\hat{X}^n(t)$ converges to $\hat{X}(t)$ in distribution. Then using the Portmanteau-Alexandrov Theorem, we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_F \sup_{a \in \mathbb{A}} \left| b_x^n(t, e^{-\alpha t} z, a) - b_x(t, e^{-\alpha t} z, a) \right|^4 \varphi_t^n(e^{-\alpha t} z) dz \\ & \leq 2\lambda_1^4(t) \limsup \tilde{P}(e^{-\alpha t} \hat{X}^n(t) \in F) \\ & \leq 2\lambda_1^4(t) \tilde{P}(e^{-\alpha t} \hat{X}(t) \in F) \\ & = 2\lambda_1^4(t) \int_F \varphi_t(e^{-\alpha t} z) dz \leq 2\lambda_1^4(t) \varepsilon. \end{aligned} \quad (\text{B.31})$$

365 We also have

$$\begin{aligned} \int_{B(0, R)} \sup_{a \in \mathbb{A}} e^{\alpha t} \left| b_x^n(t, y, a) - b_x(t, y, a) \right|^4 \varphi_t^n(y) dy &= \int_F \sup_{a \in \mathbb{A}} \left| b_x^n(t, e^{-\alpha t} z, a) - b_x(t, e^{-\alpha t} z, a) \right|^4 \varphi_t^n(e^{-\alpha t} z) dz \\ &+ \int_{F^c} \sup_{a \in \mathbb{A}} \left| b_x^n(t, e^{-\alpha t} z, a) - b_x(t, e^{-\alpha t} z, a) \right|^4 \varphi_t^n(e^{-\alpha t} z) dz \end{aligned}$$

366 Using (B.29), (B.31) and (H2), we get $\lim_{n \rightarrow \infty} I_1^n = 0$.

Let now show that $\lim_{n \rightarrow \infty} I_2^n = 0$. Let k be an integer and observe that

$$I_n^2 \leq M \left(I_{n,k}^{21} + I_{n,k}^{22} + I_{n,k}^{23} \right),$$

where M is a positive constant and

$$\begin{aligned} I_{n,k}^{21} &= \tilde{E} \left[\int_0^\infty \sup_{a \in \mathbb{A}} e^{\alpha t} \left| b_x(t, \hat{X}^n(t), a) - b_x^k(t, \hat{X}^n(t), a) \right|^4 dt \right], \\ I_{n,k}^{22} &= \tilde{E} \left[\int_0^\infty \sup_{a \in \mathbb{A}} e^{\alpha t} \left| b_x^k(t, \hat{X}^n(t), a) - b_x^k(t, \hat{X}(t), a) \right|^4 dt \right], \\ I_{n,k}^{23} &= \tilde{E} \left[\int_0^\infty \sup_{a \in \mathbb{A}} e^{\alpha t} \left| b_x^k(t, \hat{X}(t), a) - b_x(t, \hat{X}^n(t), a) \right|^4 dt \right]. \end{aligned}$$

The conclusion follows in a similar way as in [2, Lemma 3.6] since $\int_0^\infty \lambda_1(t)^4 dt < \infty$. \square

REFERENCES

- [1] K. Bahlali, B. Djehiche, B. Mezerdi (1996). The stochastic maximum principle in optimal control of a diffusion with nonsmooth coefficients. *Stoch. Stoch. Rep.*, 57: 303-316.
- [2] K. Bahlali, B. Djehiche, B. Mezerdi (2007). On the Stochastic Maximum Principle in Optimal Control of Degenerate Diffusions with Lipschitz Coefficients. *Appl Math Optim.*, 56: 364-378.
- [3] K. Bahlali, F. Chighoub, B. Djehiche, B. Mezerdi (2009). Optimality necessary conditions in singular stochastic control problems with nonsmooth data. *J. Math. Anal. Appl.*, 355 (2):479-494
- [4] A. Bensoussan (1981). Lectures on stochastic control. *Lecture Note in Mathematics*, 972: 1-62.
- [5] J. M. Bismut (1973). Conjugate convex functions in optimal stochastic control. *J. Math. Anal. Appl.*, 44: 384-404.
- [6] J. M. Bismut (1978). An introductory approach to duality in optimal stochastic control. *SIAM Review*, 20: 62-78.
- [7] N. Bouleau, F. Hirsch (1988). Sur la propriété du flot d'une équation différentielle stochastique. *C.R. Acad. Sci. Paris*, 306: 421-424.
- [8] F.H. Clarke (1983). *Optimization and Nonsmooth Analysis*. Wiley, New York.
- [9] F. Chighoub, B. Djehiche, B. Mezerdi (2009). The stochastic maximum principle in optimal control of degenerate diffusion with non-smooth coefficients. *Random Operators/ Stochastic Eqs.*, 17: 37-54.
- [10] I. Ekeland (1979). Nonconvex minimization problems. *Bulletin of the American Mathematical Society*, 1(3): 443-474.
- [11] V. H. Fleming, H. M. Soner (2006). *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag.
- [12] N. Framstad, B. Øksendal, A. Sulem (2004). Stochastic maximum principle for optimal control of jump diffusions and applications to finance. *J. Optimization Theory and Appl.* 121 (1): 77-98.
- [13] S. Haadem, B. Øksendal, F. Proske (2012). Maximum Principles for Jump Diffusions Processes with Infinite Horizon. *arXiv: 1206.1719v1*.
- [14] U. G. Hausmann (1986). *A stochastic Maximum Principle for Optimal Control of Diffusions*. Longman Scientific and Technical.
- [15] H. J. Kushner (1965). On the stochastic maximum principle: fixed time of control. *Journal of Mathematical Analysis and Applications*, 11: 78-92.
- [16] H. J. Kushner (1972). Necessary conditions for continuous parameter stochastic optimization problems. *SIAM journal on control and Optimizations*, 10: 550-565
- [17] B. Maslowski, P. Veverka (2013). Sufficient Stochastic Maximum Principle for Discounted Control Problem. *arXiv:1105.4737v2*.

- 403 [18] B. Mezerdi (1988). Necessary condition for optimality for a diffusion with non smooth drift. Sto-
404 chastic, 24: 305-326.
- 405 [19] S. Peng (1990). A general stochastic maximum principle for optimal control problems. SIAM
406 Journal of Control and Optimization, 28: 966-979.
- 407 [20] S. Peng, Y. Shi (2000). Infinite horizon forward-backward stochastic differential equations. Sto-
408 chastic processes and their applications, 85: 75-92.
- 409 [21] J. Yin (2008). On solutions of a class of infinite horizon FBSDEs. Statistic & Probability Letters,
410 78: 2412-2419.
- 411 [22] J. Yong, X. Y. Zhou (1999). Stochastic controls: Hamiltonian Systems and HJB Equa-
412 tions. Springer, New York.
- 413 [23] L. Zhang, Y. Shi (2010). Comparison Theorems of Infinite Horizon Forward-Backward Stochas-
414 tic Differential Equations. arXiv.1005.4139v1.

415 SCHOOL OF COMPUTATIONAL AND APPLIED MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE
416 BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA.
417 *E-mail address:* virginie.konlacksocgnia2@wits.ac.za

418 INSTITUTE FOR FINANCIAL AND ACTUARIAL MATHEMATICS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF
419 LIVERPOOL, LIVERPOOL, L69 7ZL, UNITED KINGDOM.
420 *E-mail address:* Menoukeu@liverpool.ac.uk